

AN EXPLICIT SKEW-SHIFT SCHRÖDINGER OPERATOR WITH POSITIVE LYAPUNOV EXPONENT AT SMALL COUPLING

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ABSTRACT. I give an example of a skew-shift Schrödinger operator with positive Lyapunov exponent in the spectrum for all positive coupling constant with constant frequency. This is the first example of this kind.

The proof is based on CMV operators given by the skew-shift. Further results on these are derived.

1. INTRODUCTION

In the theory of one dimensional ergodic Schrödinger operators $H_{g;\omega} = \Delta + gV_\omega$, there are two main regimes defined by the Lyapunov exponent

$$(1.1) \quad L_g(E) = \lim_{N \rightarrow \infty} \mathbb{E} \left(\frac{1}{N} \log \left\| \prod_{n=N}^1 \begin{pmatrix} E - gV_\omega(n) & -1 \\ 1 & 0 \end{pmatrix} \right\| \right)$$

either being positive on the spectrum or vanishing on it. For $V_\omega(n) = f(T^n\omega)$, $\omega \in \Omega$, and $T : \Omega \rightarrow \Omega$ a μ -ergodic transformation, there exists a set $\Sigma_{ac;g} \subseteq \mathbb{R}$ such that for μ almost-every ω , the absolutely continuous spectrum of $H_{g;\omega}$ is $\Sigma_{ac;g}$. We then have that $\Sigma_{ac;g}$ is equal to the essential closure of the set of energies $E \in \mathbb{R}$ such that the Lyapunov exponent $L_g(E)$ vanishes. This and related topics are usually known as *Kotani theory*, see [10].

The case of $V(n)$ being independent identically distributed random variables has been understood for some time now: the Lyapunov exponent is positive.

Much progress has been made in understanding the Lyapunov exponent for quasi-periodic Schrödinger operators, that is with potential given by $V_x(n) = f(x + n\alpha)$, where $f : \mathbb{T} \rightarrow \mathbb{R}$ is a real analytic function, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $x \in \mathbb{T}$, α irrational, $\mathbb{E}(h) = \int_{\mathbb{T}} h(x) dx$. In particular, one has a transition from vanishing Lyapunov exponent $L_g(E)$ on the spectrum for small $g > 0$ to positive Lyapunov exponent at large g .

Somewhat surprising from this, is that one expects that the Schrödinger operator with potential

$$(1.2) \quad V(n) = \cos(2\pi\omega n^2)$$

should have positive Lyapunov exponent for ω irrational and all $g > 0$, see [9] and [2, Chapter 15]. An adaptation of the methods used to show that the Lyapunov exponent is positive for large g for quasi-periodic Schrödinger operators [2, 6, 8],

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yields that the Lyapunov exponent is positive when ω is Diophantine, i.e.

$$(1.3) \quad \kappa = \inf_{\mathbb{Z} \ni q \geq 1} q^2 \|q\alpha\| > 0$$

where $\|x\| = \text{dist}(x, \mathbb{Z})$ and the largeness condition on g depends on $\kappa > 0$. The much harder and open problem is to show that the Lyapunov exponent is positive for small $g > 0$.

Potentials of the form (1.2) are of the general form

$$(1.4) \quad V_{g;\underline{x}}(n) = g \cdot f(T_\omega^n \underline{x}),$$

where $g > 0$ is a coupling constant, $\underline{x} \in \mathbb{T}^r$, $f : \mathbb{T}^r \rightarrow \mathbb{R}$, and $T_\omega : \mathbb{T}^r \rightarrow \mathbb{T}^r$ is the skew-shift given by

$$(1.5) \quad (T_\omega \underline{x})_\ell = \begin{cases} x_1 + \omega, & \ell = 1; \\ x_\ell + x_{\ell-1}, & 2 \leq \ell \leq r. \end{cases}$$

If ω obeys the Diophantine condition (1.3) and $g > 0$ is sufficiently large depending on κ , it follows from either [6] or [8] that the Lyapunov exponent is positive and Anderson localization holds in a suitable set of parameters for $r = 2$. Positive Lyapunov exponent for general $r \geq 3$ is proven in [14]. Furthermore, it is shown in [15] that the spectrum of $H_{g;\underline{x}} = \Delta + V_{g;\underline{x}}$ contains intervals for $r = 2$ and f only depending on x_2 . Although this picture is still incomplete, we have a fair amount of understanding. For example the what happens if ω is Liouville has not been investigated yet.

In the case $g > 0$ small far less is known. For $r = 2$, $f(x_1, x_2) = \cos(2\pi x_2)$, and $g > 0$ small enough, Bourgain has shown in [3, 4, 5] that for small enough ω obeying (1.3), the Lyapunov exponent is positive on a large set, which contains some spectrum. Furthermore, I have shown in [13] that for $\lambda > 0$ small enough, $r \geq 1$ large enough, and a sampling function only depending on the last coordinate x_r , the Lyapunov exponent is positive on a set of large measure. A simpler but non-quantitative proof of this result can also be found in [14, Chapter 4].

One of the objectives of this paper will be to prove

Theorem 1.1. *Let $r \geq 2$, ω Diophantine, i.e. obeying (1.3), $g > 0$, and*

$$(1.6) \quad f(\underline{x}) = \cos(2\pi x_r) - \cos(2\pi(x_r + x_{r-1})).$$

Then there exists $\varepsilon = \varepsilon(\kappa, g) > 0$ such that the Lyapunov exponent $L_g(E)$ corresponding to (1.4) satisfies

$$(1.7) \quad L_g(E) \geq \frac{1}{4} \log(1 + g^2)$$

for $|E| \leq \varepsilon$. Furthermore, 0 is contained in the spectrum of the operator.

The key difference between this theorem and the ones known so far is that, one can fix $r \geq 2$, ω obeying (1.3), and f and obtain positive Lyapunov exponent in an energy region containing some of the spectrum. Except for numerical computations of the Lyapunov exponent this is the strongest evidence so far, that we should believe in the conjecture of [9]. It should be furthermore be pointed out that the proof of Theorem 1.1 shows the correct asymptotic behavior of the Lyapunov exponent, that is $L_g(E) \sim g^2$ for g small enough (and E in a small g dependent range), and $L_g(E) \sim \log \frac{1}{g}$ for g large.

Of course, the choice of f in Theorem 1.1 should seem odd. The best explanation is that this is what comes out of the proof. Furthermore, it should be noted that the papers [3, 4, 5] require $r = 2$ and $f(x_1, x_2) = g \cos(2\pi x_2)$ to maintain a close connection to the almost Mathieu operator.

One of my hopes is that building on Theorem 1.1, further developments in the theory of skew-shift Schrödinger operators will arise. For example, it is an intriguing question how to adapt [15] to prove that the spectrum contains an interval around 0. Unfortunately, the initial estimates needed in [15] do not hold here.

Finally, let me remark that the potential $V(n) = \lambda \cos(2\pi \omega n^\rho)$ which can be thought of as interpolating between quasi-periodic and skew-shift potentials for $\rho \in (1, 2)$ can be understood, see [3, 19].

The main input into the proof is given by [17], where a certain family of orthogonal polynomials on the unit circle is analyzed by essentially algebraic tricks. The associated Verblunsky coefficients are given by

$$(1.8) \quad \alpha_{\underline{x};n} = \lambda e^{2\pi i(T^n \underline{x})_r},$$

where $\lambda \in \mathbb{D} = \{z : |z| < 1\}$. In particular, one obtains a CMV operator $\mathcal{E}_{\underline{x}}$ with Lyapunov exponent given by $-\frac{1}{2} \log(1 - |\lambda|^2)$. The main realization is now that the problem for $\mathcal{E}_{\underline{x}}$ is equivalent to a tridiagonal operator, which has constant off-diagonal terms when $z = -1$ and thus is a Schrödinger operator. The result then essentially follows by taking the real part of the CMV operator, see Corollary 2.3.

The results of [17] are by themselves not strong enough yet to imply Theorem 1.1. One further goal of this paper is to improve this results, in particular to show that exponential decay of the Green's function holds with super polynomially small probability and thus one has as good results as in the Schrödinger case.

This brings me to the second motivation for writing this paper. It came as a surprise that the microscopic eigenvalue statistics in the case $r = 2$ is very regular as shown in [17]. I have recently shown in [18] that this is not the case for Schrödinger operators. The results of this paper allow one to extend this to CMV operators as discussed in [17] when $r \geq 2$ and in particular one sees that the microscopic distribution of the eigenvalues is much closer to the one of the Anderson model than the one in the case $r = 2$.

The content of the rest of the paper can be described as follows. In Section 2, I discuss the background on CMV operators necessary for the proof of Theorem 1.1. In particular, in Corollary 2.3 a weak version of Theorem 1.1 is derived. Furthermore, Theorem 2.5 and Theorem 2.6 present results on CMV operators, which are of independent interest.

Section 3 contains the proof of Theorem 1.1 and further discussions of properties of this Schrödinger operator. Section 4 proves facts about the semi-algebraic structure of certain sets related to Green's function estimates for CMV operators. Section 5 contains the proof of Theorem 2.5, which is the biggest technical progress given in this paper. Finally, Appendix A recalls some facts about the return times of the skew-shift to a semi-algebraic set.

2. CMV OPERATORS

In this section, we introduce large parts of the notation necessary to prove Theorem 1.1, in particular CMV operators. As much of the notation related to CMV operators is similar to the one of Schrödinger operators, I will defer the proof of Theorem 1.1 to the next section.

Let us now begin by introducing the necessary notation. Given a bi-infinite sequence of Verblunsky coefficients $\alpha_n \in \mathbb{D} = \{z : |z| < 1\}$, we define the matrices

$$(2.1) \quad \Theta_n = \begin{pmatrix} \overline{\alpha_n} & \rho_n \\ \rho_n & -\alpha_n \end{pmatrix}, \quad \rho_n = \sqrt{1 - |\alpha_n|^2}$$

thought of as acting on $\ell^2(\{n, n+1\})$. Then, we define the operators

$$(2.2) \quad \mathcal{L} = \bigoplus_{n \text{ even}} \Theta_n, \quad \mathcal{M} = \bigoplus_{n \text{ odd}} \Theta_n$$

and the extended CMV operator $\mathcal{E} = \mathcal{L} \cdot \mathcal{M}$. One can easily check that $\mathcal{E} : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ is an unitary operator. CMV operators are usually discussed in the context of orthogonal polynomials on the unit circle, when it is more natural to consider half line objects. For our purposes the whole line operator is more natural. The orthogonal polynomial aspects of the theory have been worked out extensively, see for example the books [20, 21] and the extensive references therein.

For $\beta, \tilde{\beta} \in \partial\mathbb{D} = \{z : |z| = 1\}$ and $a < b \in \mathbb{Z}$, the restriction $\mathcal{L}_{\beta, \tilde{\beta}}^{[a, b]}$ is defined by setting $\alpha_{a-1} = \beta$, $\alpha_b = \tilde{\beta}$ and restricting the resulting CMV operator to $\ell^2([a, b])$. $\mathcal{M}_{\beta, \tilde{\beta}}^{[a, b]}$ is defined in a similar way. Finally, one defines

$$(2.3) \quad \mathcal{E}_{\beta, \tilde{\beta}}^{[a, b]} = \mathcal{L}_{\beta, \tilde{\beta}}^{[a, b]} \cdot \mathcal{M}_{\beta, \tilde{\beta}}^{[a, b]}.$$

All these operators are unitary. Furthermore, it should be noted that β and $\tilde{\beta}$ take the role of boundary conditions. To understand the content of the next lemma, it is important to observe that $(z - \mathcal{E})\psi = 0$ is equivalent to $(z\mathcal{L}^* - \mathcal{M})\psi = 0$ as \mathcal{L} is unitary.

Lemma 2.1. *The matrix $A = z(\mathcal{L}_{\beta, \tilde{\beta}}^{[a, b]})^* - \mathcal{M}_{\beta, \tilde{\beta}}^{[a, b]}$ is tridiagonal. Write $A = \{A_{i,j}\}_{a \leq i,j \leq b}$. Then we have that*

$$(2.4) \quad A_{j,j} = \begin{cases} z\alpha_j + \alpha_{j-1}, & j \text{ even}; \\ -z\overline{\alpha_{j-1}} - \overline{\alpha_j}, & j \text{ odd}, \end{cases} \quad A_{j+1,j} = A_{j,j+1} = \tilde{\rho}_j = \begin{cases} z\rho_j, & j \text{ even}; \\ -\rho_j, & j \text{ odd}. \end{cases}$$

By Lemma 2.1, we have with $z = -1$ that the matrix of the operator $B = \text{Re}(z(\mathcal{L}_{\beta, \tilde{\beta}}^{[a, b]})^* - \mathcal{M}_{\beta, \tilde{\beta}}^{[a, b]})$ are given by

$$(2.5) \quad B_{j,j} = \text{Re}(\alpha_{j-1} - \alpha_j), \quad B_{j,j+1} = B_{j+1,j} = -\rho_j.$$

Let us now discuss the Verblunsky coefficients $\alpha_{\underline{x};n} = \lambda \exp(2\pi i(T_\omega^n \underline{x})_r)$, where $T_\omega : \mathbb{T}^r \rightarrow \mathbb{T}^r$ is the skew-shift with frequency ω . Then, we have that $\rho_j = \sqrt{1 - |\lambda|^2}$ and that the diagonal elements are given by

$$(2.6) \quad B_{j,j} = V_{\underline{x}}(j) = \text{Re}(\lambda(\exp(2\pi i(T_\omega^{j-1} \underline{x})_r) - \exp(2\pi i(T_\omega^j \underline{x})_r))).$$

This clearly has the form of a skew-shift potential.

Theorem 2.2. *Let $r \geq 2$ and $\lambda \in \mathbb{D} \setminus \{0\}$. We have that $\sigma(\mathcal{E}_{\underline{x}})$ and for $z \in \partial\mathbb{D}$ that*

$$(2.7) \quad \ell_\lambda(z) = -\frac{1}{2} \log(1 - |\lambda|^2).$$

Proof. This is in [17]. \square

Here ℓ_λ denotes the Lyapunov exponent for CMV operators given by

$$(2.8) \quad \ell_\lambda(z) = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left(\log \left\| \prod_{n=1}^N \frac{1}{\rho_{\underline{x};n}} \begin{pmatrix} z & -\overline{\alpha_{\underline{x};n}} \\ -z\alpha_{\underline{x};n} & 1 \end{pmatrix} \right\| \right),$$

where $\mathbb{E}(h) = \int_{\mathbb{T}^r} h(\underline{x}) d\underline{x}$. This theorem combined with the considerations preceding it imply the following weak version of Theorem 1.1.

Corollary 2.3. *Let $H_{g;\underline{x}}$ be the Schrödinger operator with potential as above. Then 0 is in the spectrum and the Lyapunov exponent is given by*

$$(2.9) \quad L_g(0) = \frac{1}{2} \log(1 + g^2).$$

Proof. We first note that

$$-\frac{1}{\sqrt{1-|\lambda|^2}}(-\mathcal{L}_{\underline{x}}^* - \mathcal{M}_{\underline{x}}) = \Delta + V,$$

where V is given by (1.4) and

$$f(\underline{x}) = \operatorname{Re} \left(e^{2\pi i (T_\omega^{-1} \underline{x})_r} - e^{2\pi i (\underline{x})_r} \right) = \cos(2\pi (T_\omega^{-1} \underline{x})_r) - \cos(2\pi x_r), \quad g = \frac{\lambda}{\sqrt{1-|\lambda|^2}}.$$

That $0 \in \sigma(H_{g;\underline{x}})$ follows from spectral calculus. To see the claim about the Lyapunov exponent, observe that for almost every $\underline{x} \in \mathbb{T}^r$ there exists ψ such that

$$(-\mathcal{L}_{\underline{x}}^* - \mathcal{M}_{\underline{x}})\psi = 0$$

and $\frac{1}{2n} \log(|\psi(n)|^2 + |\psi(n+1)|^2) \rightarrow \frac{1}{2} \log(1 - |\lambda|^2)$. As ψ also solves $H_{g;\underline{x}}\psi = 0$, the claim follows. \square

The main problem in extending this to prove Theorem 1.1 is to achieve stability in the energy, we will do this by instead proving a finite scale claim about the Green's function. Let $z \in \mathbb{C}$, $\beta, \tilde{\beta} \in \partial\mathbb{D}$, $a \leq k, \ell \leq b$, then the Green's function is defined by

$$(2.10) \quad G_{\beta, \tilde{\beta}}^{[a,b]}(z; k, \ell) = \left\langle \delta_k, \left(z(\mathcal{L}_{\beta, \tilde{\beta}}^{[a,b]})^* - \mathcal{M}_{\beta, \tilde{\beta}}^{[a,b]} \right)^{-1} \delta_\ell \right\rangle.$$

As discussed in [17], the advantage over considering the matrix elements of $(z - \mathcal{E}_{\beta, \tilde{\beta}}^{[a,b]})^{-1}$ is that an application of Cramer's rule yields much simpler terms. By Lemma 3.9 in [17], we have

$$(2.11) \quad \begin{aligned} |\psi(n)| &\leq 2|G_{\beta, \tilde{\beta}}^{[a,b]}(z; n, a)| \sup_{m \in \{a-1, a\}} |\psi(m)| \\ &\quad + 2|G_{\beta, \tilde{\beta}}^{[a,b]}(z; n, b)| \sup_{m \in \{b, b+1\}} |\psi(m)| \end{aligned}$$

if ψ solves $\mathcal{E}\psi = z\psi$, $a < n < b$, and $\beta, \tilde{\beta} \in \partial\mathbb{D}$.

We describe the decay properties of the Green's function using the following definition.

Definition 2.4. Let $[-N, N] \subseteq \mathbb{Z}$ be an interval, $z, \beta, \tilde{\beta} \in \partial\mathbb{D}$, $\Gamma > 0$, $\gamma > 0$, $p \geq 0$. Then $[a, b]$ is called (γ, Γ, p) -suitable for $\mathcal{E}_{\beta, \tilde{\beta}} - z$, if

$$(2.12) \quad \begin{aligned} & \text{(i) } \|(\mathcal{E}_{\beta, \tilde{\beta}}^{[-N, N]} - z)^{-1}\| \leq \frac{1}{2^p} e^\Gamma. \\ & \text{(ii) For } k, \ell \in [-N, N] \text{ with } |k - \ell| \geq \frac{N}{2}, \text{ we have} \\ & |G_{\beta, \tilde{\beta}}^{[a, b]}(z; k, \ell)| \leq \frac{1}{2^{p+1}} e^{-\gamma|k-\ell|}. \end{aligned}$$

We are now ready for

Theorem 2.5. Let $\lambda \in \mathbb{D} \setminus \{0\}$, $z, \beta, \tilde{\beta} \in \partial\mathbb{D}$, and assume ω obeys (1.3). Consider the CMV operator with Verblunsky coefficients given by

$$(2.13) \quad \alpha_{\underline{x}; n} = \lambda \exp(2\pi i (T_\omega^n \underline{x})_r).$$

There exist constants $\sigma \in (0, 1)$, $\tau \in (0, 1)$, and $\gamma > 0$ such that for N sufficiently large

$$(2.14) \quad |\{\underline{x} \in \mathbb{T}^r : [-N, N] \text{ is } (\gamma, N^\tau, p)\text{-unsuitable for } \mathcal{E}_{\beta, \tilde{\beta}} - z\}| \leq e^{-N^\sigma}.$$

Once, this theorem is established proving Theorem 1.1 can be done by standard methods as discussed in [14]. We will discuss this in detail in the next section. The proof of Theorem 2.5 is given in Sections 4 and 5.

I will now explain some more consequences for CMV operators of Theorem 2.5. The following theorem is essentially just an application of appropriate perturbation theory. For a simplified statement, we define

$$(2.15) \quad f_0(\underline{x}) = \lambda \exp(2\pi i (\underline{x})_r)$$

and the $w > 0$ neighborhood of \mathbb{R}^r by

$$(2.16) \quad \mathcal{A}_w = \{\underline{z} \in \mathbb{C}^r : |\operatorname{Im}(z_j)| \leq w\}.$$

A function $f : \mathcal{A}_w \rightarrow \mathbb{C}$ is one periodic if $f(\underline{z} + \underline{n}) = f(\underline{z})$ for all $\underline{n} \in \mathbb{Z}^r$ and $\underline{z} \in \mathcal{A}_w$.

Theorem 2.6. Let $r \geq 2$, $w > 0$, $\kappa > 0$, ω such that (1.3) holds. Then there exist $\gamma = \gamma(\kappa, r, w, \lambda) > 0$, and $\varepsilon_0 = \varepsilon_0(\kappa, r, w, \lambda) > 0$ such that if for $f : \mathcal{A}_w \rightarrow \mathbb{C}$ analytic and one periodic with

$$(2.17) \quad \|f - f_0\|_w = \sup_{|\operatorname{Im}(x_j)| \leq w} |f(\underline{x}) - f_0(\underline{x})| < \varepsilon_0,$$

we have that there exists a set \mathcal{B} such that the conclusions of Theorem 2.5 hold for the Verblunsky coefficients

$$(2.18) \quad \alpha_{\underline{x}; n} = f(T_\omega^n \underline{x}).$$

In order to give a proof of this result, we first need

Lemma 2.7. Let $[a, b]$ be (γ, Γ, p) -suitable for $\mathcal{E}_{\beta, \tilde{\beta}} - z$. Then if

$$(2.19) \quad \sup_{n \in [a, b]} |\hat{\alpha}_n - \alpha_n|, |\hat{z} - z| \leq e^{-(2\Gamma + \gamma|b-a|)}$$

we have

$$(2.20) \quad [a, b] \text{ is } (\gamma, \Gamma, p-1)\text{-suitable for } \hat{\mathcal{E}}_{\beta, \tilde{\beta}} - \hat{z}.$$

Proof. This is just an application of the resolvent equation and some estimates. \square

Proof of Theorem 2.6. The previous lemma implies that the conclusions for fixed N of Theorem 2.5 are stable under small perturbations of f in the $\|\cdot\|_w$ topology. Then using the results of Sections 4 and 5, it is possible to adopt the results of [14] to show that the conclusion of Theorem 2.5 for $1 \leq N \leq N_0$ for some large enough N_0 implies them for all $N \geq 1$. \square

Furthermore, using the results of [2, Chapter 15], it is possible to show Anderson and dynamical localization for these CMV operators as long as $r = 2$.

3. PROOF OF THEOREM 1.1

As $-\log(1-t) \geq t$ for $t > 0$, if we establish that the Lyapunov exponent is continuous, we obtain that

$$(3.1) \quad L_g(E) \geq \frac{g^2}{4}$$

for $g > 0$ small and E small by Corollary 2.3. Of course, establishing continuity of the Lyapunov exponent will be a non-trivial task. Our proof will essentially follow the ideas of [14].

For $H = \Delta + V : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ and $[a, b] \subseteq \mathbb{Z}$ an interval, we define $H^{[a, b]}$ to be the restriction of H to $\ell^2([a, b])$. As H is self-adjoint also $H^{[a, b]}$ is. For $z \in \mathbb{C}$, $k, \ell \in [a, b]$, we define the *Green's function* by

$$(3.2) \quad G^{[a, b]}(z; k, \ell) = \left\langle \delta_k, (H^{[a, b]} - z)^{-1} \delta_\ell \right\rangle.$$

As in the case of CMV operators, we could make all the definitions with boundary conditions. However, I have decided not to do so, as it isn't necessary to have self-adjoint operators, and to have different notations for CMV and Schrödinger operators. It is clear how to adapt Definition 2.4 into this new context. Furthermore, Theorem 2.5 can be seen to imply that

Theorem 3.1. *Let $\lambda > 0$ and ω obey (1.3). Then there exist constants $\sigma, \tau \in (0, 1)$, $\gamma > 0$ such that for the Schrödinger operator with potential given by (1.4) and (1.6), we have*

$$(3.3) \quad |\{\underline{x} \in \mathbb{T}^r : [-N, N] \text{ is not } (\gamma, N^\tau, 3)\text{-suitable for } H_{g; \underline{x}}\}| \leq e^{-N^\sigma}$$

for $N \geq 1$ sufficiently large.

Proof. By construction, we have as in Corollary 2.3 that

$$H_{g; \underline{x}} = \frac{1}{\sqrt{1 - \lambda^2}} \operatorname{Re}(-\mathcal{L}_{\underline{x}} - \mathcal{M}_{\underline{x}}).$$

So the result follows. \square

Of course, this theorem only holds for $E = 0$. Let N be such that the conclusion holds. Then we see from an easy perturbation argument as in the proof of Theorem 2.6 that we have for $|E| \leq e^{-3N}$ that

$$(3.4) \quad |\{\underline{x} \in \mathbb{T}^r : [-N, N] \text{ is not } (\gamma, N^\tau, 2)\text{-suitable for } H_{g; \underline{x}} - E\}| \leq e^{-N^\sigma}.$$

From this, we can conclude

Theorem 3.2. *There exists $\varepsilon > 0$ such that for $|E| \leq \varepsilon$, we have that (3.4) holds for $N \geq 1$ large enough.*

Proof. This follows by an iterative application of Theorem 7.4 in [14]. Alternatively, one can also adapt the methods of Chapter 15 in [2] to prove this result. \square

This theorem implies by itself that the Lyapunov exponent is positive. Unfortunately, it does not imply the correct size of the Lyapunov exponent, as $\gamma > 0$ is much too small. This is the reason, we conclude the claim of positivity of the Lyapunov exponent from continuity.

The integrated density of states is defined by

$$(3.5) \quad k(E) = \lim_{N \rightarrow \infty} \frac{1}{N} \int_{\mathbb{T}^r} \#\{\text{eigenvalues of } H_{g;\underline{x}}^{[1,N]} \leq E\} d\underline{x}.$$

Corollary 7.6. in [14] implies that there exists some $c > 0$ such that

$$(3.6) \quad k(E + \delta) - k(E - \delta) \leq \exp\left(-\log\left(\frac{1}{\delta}\right)^c\right)$$

for $|E| \leq \varepsilon$ and $\delta > 0$ small enough. As the Lyapunov exponent satisfies

$$(3.7) \quad L(E) = \int \log |E - t| dk(t),$$

we have that for $|E| \leq \varepsilon$ the Lyapunov exponent has similar continuity properties, and thus is continuous, which implies our main claim. The relation between continuity of the Lyapunov exponent and the integrated density of states is discussed in Section 10 of [11].

Finally, let me remark that in case $r = 2$, the results of Chapter 15 in [2] imply that Anderson localization holds. We take this to mean that the spectrum of $H_{g;\underline{x}}$ is pure point in $[-\varepsilon, \varepsilon]$ and that the corresponding eigenfunctions ψ_ℓ satisfy

$$(3.8) \quad |\psi_\ell(n)| \leq C_\ell e^{-\gamma|n|}$$

for some $C_\ell > 0$ and a uniform constant $\gamma > 0$.

4. SEMIALGEBRAIC STRUCTURE OF SUITABILITY

In this section, we investigate the geometric structure of the set of suitability for CMV operators. We will classify how complicated a set is by its semi-algebraic description. That is to say a set is simple if it can be described by few polynomial equations involving polynomials of low degree. Some background on these questions can be found in Chapter 9 of [2]. These methods were first introduced in [7]. For general background see [1].

A set $\mathcal{S} \subseteq \mathbb{T}^K$ is a semi-algebraic set if there exist polynomials $P_1, \dots, P_b, Q_1, \dots, Q_B \in \mathbb{R}[x_1, \dots, x_K]$ such that

$$(4.1) \quad \mathcal{S} = \bigcup_{j=1}^b \{\underline{x} : P_j(\underline{x}) \leq 0\} \cup \bigcup_{j=1}^B \{\underline{x} : Q_j(\underline{x}) = 0\}.$$

The degree of the semi-algebraic set \mathcal{S} is given by

$$(4.2) \quad \deg(\mathcal{S}) \leq (b + B) \cdot \max(\deg(P_j), \deg(Q_j)),$$

where the degree is the minimal value of the left-hand side.

The goal of this section is to show

Theorem 4.1. *Let $\Gamma > 0$, $\gamma_\infty > 0$, $N \geq 1$ large enough, $|a|, |b| \leq N$, $\gamma_{k,\ell} \in [0, \gamma_\infty \cdot N]$ for $k, \ell \in [a, b]$, and $\beta, \tilde{\beta} \in \partial\mathbb{D}$. Let $\mathcal{U}_p \subseteq \mathbb{T}^r$ be a set of \underline{x} such that*

$$(4.3) \quad \|(z(\mathcal{L}_{\underline{x};\beta,\tilde{\beta}}^{[a,b]})^* - \mathcal{M}_{\underline{x};\beta,\tilde{\beta}}^{[a,b]})^{-1}\| \leq \frac{1}{2^{p+1}} e^\Gamma$$

and for $k, \ell \in [a, b]$ with $\gamma_{k,\ell} > 0$

$$(4.4) \quad |G_{\underline{x};\beta,\tilde{\beta}}^{[a,b]}(z; k, \ell)| \leq \frac{1}{2^p} e^{-\gamma_{k,\ell}}.$$

Then there exists a semi-algebraic set U of degree N^B such that

$$(4.5) \quad \mathcal{U}_{p-1} \subseteq U \subseteq \mathcal{U}_{p+1},$$

where $B \geq 1$ is some universal constant.

How to apply this theorem can be seen in combination with Theorem A.3, once one knows a measure estimate on \mathcal{U}_{p+1} . We now begin the proof of Theorem 4.1.

We recall that for us $f : \mathbb{R}^r \rightarrow \mathbb{C}$ is a real-analytic function, that is there exists an analytic extension of f to a neighborhood of the form

$$(4.6) \quad \mathcal{A}_w = \{\underline{z} \in \mathbb{C}^r : |\operatorname{Im}(z_j)| \leq w, j = 1, \dots, r\},$$

where $w > 0$. Such an f is one periodic if

$$(4.7) \quad f(\underline{z} + \underline{n}) = f(\underline{z})$$

for all $\underline{z} \in \mathcal{A}_w$ and $\underline{n} \in \mathbb{Z}^r$. We define the following norm on these functions

$$(4.8) \quad \|f\|_w = \sup_{\underline{z} \in \mathcal{A}_w} |f(\underline{z})|.$$

We furthermore recall that $g : \mathbb{R}^r \rightarrow \mathbb{R}$ is a *trigonometric polynomial* of degree d if it can be written in the form

$$(4.9) \quad g(\underline{x}) = \sum_{\underline{k} \in \mathbb{Z}^r, |\underline{k}|_\infty \leq d} \hat{g}(\underline{k}) e(\underline{k} \cdot \underline{x})$$

where $|\underline{k}|_\infty = \sup_{1 \leq j \leq r} |k_j|$, $e(x) = e^{2\pi i x}$, and $\underline{k} \cdot \underline{x} = \sum_j k_j x_j$. One easily sees that a trigonometric polynomial is a one-periodic function $\mathcal{A}_w \rightarrow \mathbb{C}$ for any $w > 0$.

Define

$$(4.10) \quad \begin{aligned} \tilde{T} : \mathbb{R}^r &\rightarrow \mathbb{R}^r, \\ (\tilde{T}\underline{x})_\ell &= \begin{cases} x_1 + \omega, & \ell = 1; \\ x_\ell + x_{\ell-1}, & 2 \leq \ell \leq r. \end{cases} \end{aligned}$$

Then the skew-shift is just $T\underline{x} = \tilde{T}\underline{x} \pmod{1}$. In particular, we have that

$$(4.11) \quad \alpha_{\underline{x};n} = f(\tilde{T}^n \underline{x}), \quad \rho_{\underline{x};n} = \sqrt{1 - |f(\tilde{T}^n \underline{x})|^2}$$

which is defined as long as $|f(\tilde{T}^n \underline{x})| \leq 1$.

Lemma 4.2. *Let $n \in \mathbb{Z}$. There exist trigonometric polynomials p_1 and p_2 of degree $n^{r-1}D$ such that*

$$(4.12) \quad |\alpha_{\underline{x};n} - p_1(\underline{x})| \leq C e^{-\frac{\rho}{2}D}, \quad |\rho_{\underline{x};n} - p_2(\underline{x})| \leq C e^{-\frac{\rho}{2}\sqrt{D}}$$

for $\underline{x} \in \mathbb{R}^r$ and some universal constant $C > 0$.

Proof of Lemma 4.2 for $\alpha_{\underline{x};n}$. One can easily check that the Fourier coefficients of f satisfy $|\hat{f}(\underline{k})| \leq \|f\|_\rho e^{-\rho|\underline{k}|^\infty}$. Introduce

$$\tilde{p}_1(\underline{x}) = \sum_{|\underline{k}| \leq D} \hat{f}(\underline{k}) e(\underline{k} \cdot \underline{x})$$

and $p_1(\underline{x}) = p_1(\tilde{T}^n \underline{x})$. As the components $\tilde{T}^n \underline{x}$ are polynomials of degree at most n^{r-1} , the claim follows. \square

Even when $\alpha_{\underline{x};n}$ is a polynomial in \underline{x} , it is not clear that $\rho_{\underline{x};n} = \sqrt{1 - |\alpha_{\underline{x};n}|^2}$ is. We first note the Taylor series expansion

$$(4.13) \quad \sqrt{1-x} = \sum_{n=0}^{\infty} \frac{(2n)!}{(1-2n)(n!)^2 4^n} x^n,$$

which converges for $|x| \leq 1$. As $\left| \frac{(2n)!}{(1-2n)(n!)^2 4^n} \right| \leq 1$, we obtain for $r \in (0, 1)$ the error estimate

$$(4.14) \quad \left| \sqrt{1-x} - \sum_{n=0}^N \frac{(2n)!}{(1-2n)(n!)^2 4^n} x^n \right| \leq \frac{r^N}{1-r}$$

for $|x| \leq r$ and $N \geq 1$.

Proof of Lemma 4.2 for $\rho_{\underline{x};n}$. There is some $r \in (0, 1)$ such that $|f(\underline{x})| \leq r$ for all $\underline{x} \in \mathbb{T}^r$. By Lemma 4.2 with $\lfloor D^{\frac{1}{2}} \rfloor$ in place of D , we thus obtain a trigonometric polynomial $p_1(\underline{x})$ such that

$$|p_1(\underline{x})| \leq \frac{1+r}{2} \in (0, 1)$$

for $\underline{x} \in \mathbb{T}^r$. Defining

$$p_2(\underline{x}) = \sum_{n=0}^{\lfloor D^{\frac{1}{2}} \rfloor} \frac{(2n)!}{(1-2n)(n!)^2 4^n} (p_1(\underline{x}))^n$$

we obtain a trigonometric polynomial of degree $\leq D$ that satisfies

$$|p_2(\underline{x}) - \rho_{\underline{x};n}| \lesssim e^{-c\sqrt{D}}$$

for some $c > 0$ by (4.14). \square

Our main conclusion from Lemma 4.2 is

Corollary 4.3. *There exists a constant $B > 0$ such that for $N \geq 1$ large enough, $\beta, \tilde{\beta} \in \partial D$, and $|a|, |b| \leq N$, there exists a matrix $A(\underline{x})$ whose entries are polynomials of degree less than N^B such that*

$$(4.15) \quad \|(z(\mathcal{L}_{\underline{x};\beta,\tilde{\beta}}^{[a,b]})^* - \mathcal{M}_{\underline{x};\beta,\tilde{\beta}}^{[a,b]}) - A(\underline{x})\| \leq e^{-N^2}.$$

Given a matrix $A \in \mathbb{C}^{N \times N}$, we define its Hilbert–Schmidt norm by

$$(4.16) \quad \|A\|_{\text{HS}} = \left(\sum_{n=1}^N \sum_{m=1}^N |A_{n,m}|^2 \right)^{\frac{1}{2}}.$$

It is well known that we have $\|A\| \leq \|A\|_{\text{HS}} \leq \sqrt{N} \|A\|$. Furthermore, if the entries of A are polynomials of degree at most d , then $(\|A\|_{\text{HS}})^2$ is a polynomial of degree at most $2d$.

Lemma 4.4. *Let $A(\underline{x})$ be a $n \times n$ matrix whose entries are trigonometric polynomials of degree d . Then the condition*

$$(4.17) \quad \|A(\underline{x})^{-1}\|_{\text{HS}} \leq C$$

defines a semi-algebraic set of degree $\leq 2nd$ and the condition

$$(4.18) \quad |\langle \delta_k, A(\underline{x})^{-1} \delta_\ell \rangle| \leq C$$

a semi-algebraic set of degree $\leq 2nd$.

Proof. $\det(A(\underline{x}))$ is a trigonometric polynomial of degree at most nd . As the first condition is equivalent to

$$\sum_{\text{minors } \tilde{A} \text{ of } A} \det(\tilde{A})^2 \leq C \det(A)^2$$

the claim follows. The second claim is similar. \square

Proof of Theorem 4.1. By Corollary 4.3, there exists $A(\underline{x})$ such that

$$\|(z(\mathcal{L}_{\underline{x}; \beta, \tilde{\beta}}^{[a,b]})^* - \mathcal{M}_{\underline{x}; \beta, \tilde{\beta}}^{[a,b]}) - A(\underline{x})\| \leq e^{-N^2}.$$

Define the set U as the set of all \underline{x} such that

$$\|A(\underline{x})^{-1}\| \leq \frac{1}{2^{p+1}} e^\Gamma$$

and for k, ℓ with $\gamma_{k, \ell} \neq 0$, we have

$$|\langle \delta_k, A(\underline{x})^{-1} \delta_\ell \rangle| \leq \frac{1}{2^p} e^{-\gamma_{k, \ell}}.$$

By the previous lemma, this is a semi-algebraic set of the right degree.

By Lemma 2.7, we have the right inclusions with the sets \mathcal{U}_{p-1} and \mathcal{U}_{p+1} . \square

5. PROOF OF THEOREM 2.5

The goal of this section is to prove Theorem 2.5. Before beginning, with the proof, we will need to discuss some additional facts about CMV operators. The basics can be found in Section 2.

Lemma 5.1. *Let $z_0 \in \partial\mathbb{D}$, $\beta, \tilde{\beta} \in \partial\mathbb{D}$, and $\text{dist}(z_0, \sigma(\mathcal{E}_{\beta, \tilde{\beta}}^{[a,b]})) \geq \delta$. Then for $k, \ell \in [a, b]$*

$$(5.1) \quad |G_{\beta, \tilde{\beta}}^{[a,b]}(z_0; k, \ell)| \leq \frac{1}{\delta}.$$

Proof. By assumption $\|(\mathcal{E}_{\beta, \tilde{\beta}}^{[a,b]} - z_0)^{-1}\| \leq \frac{1}{\delta}$ and $z(\mathcal{L}_{\beta, \tilde{\beta}}^{[a,b]})^* - \mathcal{M}_{\beta, \tilde{\beta}}^{[a,b]} = (\mathcal{L}_{\beta, \tilde{\beta}}^{[a,b]})^* \cdot (z - \mathcal{E}_{\beta, \tilde{\beta}}^{[a,b]})$. By $\|(\mathcal{L}_{\beta, \tilde{\beta}}^{[a,b]})^*\| = 1$, the claim follows. \square

We will need the following lemma relating the Green's function on a large scale $[a, b]$ to the Green's function on a small scale $[c, d] \subseteq [a, b]$.

Lemma 5.2. *Let $[c, d] \subseteq [a, b]$ be finite intervals in \mathbb{Z} , $\beta, \tilde{\beta}, \gamma, \tilde{\gamma} \in \partial\mathbb{D}$, $z \in \partial\mathbb{D}$, and $k \in [c, d]$, $\ell \in [a, b] \setminus [c, d]$. Then*

$$(5.2) \quad |G_{\beta, \tilde{\beta}}^{[a,b]}(z; k, \ell)| \lesssim \sup_{n \in \{c, c-1, c+1, d, d-1, d+1\}} |G_{\beta, \tilde{\beta}}^{[a,b]}(z; k, n)| \cdot \sup_{n \in \{c, c+1, d, d-1\}} |G_{\gamma, \tilde{\gamma}}^{[c,d]}(z; n, \ell)|$$

Proof. Let $A = z(\mathcal{L}_{\beta, \tilde{\beta}}^{[a, b]})^* - \mathcal{M}_{\beta, \tilde{\beta}}^{[a, b]}$, $B = z(\mathcal{L}_{\gamma, \tilde{\gamma}}^{[c, d]})^* - \mathcal{M}_{\gamma, \tilde{\gamma}}^{[c, d]}$. Denote by A_1 the restriction of A to $\ell^2([a, b] \setminus [c, d])$ and $\Gamma = A_1 \oplus B - A$. Note A, B are unitary, but A_1, Γ are not.

By the resolvent equation, we have that

$$A^{-1} - (A_1 \oplus B)^{-1} = A^{-1} \Gamma (A_1 \oplus B)^{-1}.$$

As clearly $\langle \delta_k, (A_1 \oplus B)^{-1} \delta_\ell \rangle = 0$, we obtain the claim by computing Γ and collecting terms. \square

Here and in the following, we use the notation $A \lesssim B$ for that there exists a universal constant $C > 0$ such that $A \leq CB$. Of course, the previous lemma also holds for perturbing the boundary conditions $\beta, \tilde{\beta}$. We omit this in the statement to avoid awkward notation.

We now return to the proof of Theorem 2.5 by recalling some results from [17] and reformulating them in a way that will be useful to us.

Lemma 5.3. *Let $\beta, \tilde{\beta}, z \in \partial\mathbb{D}$, $N \geq 1$, and $\varepsilon > 0$.*

Then there exists a set $\mathcal{B}_N^{IDS} \subseteq \mathbb{T}^r$ of measure

$$(5.3) \quad |\mathcal{B}_N^{IDS}| \leq \varepsilon$$

such that for all $\underline{x} \in \mathbb{T}^r \setminus \mathcal{B}_N^{IDS}$ and $k, \ell \in [-N, N]$

$$(5.4) \quad \|(\mathcal{E}_{\underline{x}; \beta, \tilde{\beta}}^{[k, \ell]} - z)^{-1}\| \lesssim \frac{N^3}{\varepsilon}.$$

Proof. By the Wegner estimate, Theorem 5.3. in [17], we have for each choice of $k, \ell \in [-N, N]$ that

$$|\{\underline{x} : \|(\mathcal{E}_{\underline{x}; \beta, \tilde{\beta}}^{[k, \ell]} - x)^{-1}\| > B\}| \lesssim \frac{N}{B}.$$

Choosing B such that the right hand side is $\frac{\varepsilon}{2N^2}$ and taking the union over all possible choices of k, ℓ , the claim follows (there are $N(2N-1)$ many choices of k and ℓ). \square

Next, we have the following result for the Green's function

Proposition 5.4. *Let $z \in \partial\mathbb{D}$ and $\eta > 1$, $N \geq 1$ large enough depending on r . Then there exists a set $\mathcal{B}_N^1 \subseteq \mathbb{T}^r$ such that*

$$(5.5) \quad |\mathcal{B}_N^1| \leq \frac{1}{N^{\frac{\eta}{2}}}$$

and for $\underline{x} \in \mathbb{T}^r \setminus \mathcal{B}_N^1$ we have that there exist $\beta, \tilde{\beta} \in \mathbb{T}$ such that

$$(5.6) \quad \|(\mathcal{E}_{\underline{x}; \beta, \tilde{\beta}}^{[-N, N]} - z)^{-1}\| \leq N^\eta$$

and for $|\ell| \leq \frac{N}{2}$ we have

$$(5.7) \quad |G_{\underline{x}; \beta, \tilde{\beta}}^{[-N, N]}(z; \ell, \pm N)| \leq \frac{1}{N^\eta}.$$

Proof. Use [17, Theorem 5.1.] to obtain the necessary estimates on the resolvent and [17, Theorem 7.1.] to obtain the decay on the Green's function. The results of [17] are only stated in the case $r = 2$, but using the results from Appendix A, it is clear that they extend to $r \geq 3$. \square

The next lemma draws a conclusion from Proposition 5.4, which allows us to use methods similar to multiscale analysis to prove our main claim. The main idea is to trade probability for better decay of the Green's function in space.

Lemma 5.5. *Let $\eta > 1$. There exists $c \in (0, 1)$ such that for $N \geq 1$ large enough there exists a set $\mathcal{B} \subseteq \mathbb{T}^r$ of measure*

$$(5.8) \quad |\mathcal{B}| \leq \frac{1}{N^{\frac{\eta}{4}}}$$

such that $\underline{x} \in \mathbb{T}^r \setminus \mathcal{B}$, we have for $k \in \{-N, N\}$ and $|\ell| \leq \frac{N}{2}$

$$(5.9) \quad |G_{\underline{x}; \beta, \tilde{\beta}}^{[-N, N]}(z; k, \ell)| \leq e^{-\gamma|k-\ell|}$$

where $\gamma = \frac{1}{N^c}$.

Proof. We denote by \mathcal{B}_M^1 the bad set from Proposition 5.4. Define

$$\mathcal{B}_{M;L}^1 = \bigcup_{\ell=-L}^L T^{\ell \lfloor \frac{M}{2} \rfloor} \mathcal{B}_M^1.$$

As the skew-shift is measure preserving, we have that $|\mathcal{B}_{M;L}^1| \leq (2L+1)|\mathcal{B}_M^1|$. So taking $L = \lfloor \frac{1}{3}M^{\frac{\eta}{4}} \rfloor$, we obtain a set $\mathcal{B}_L^2 = \mathcal{B}_{M;L}^1$ with measure

$$|\mathcal{B}_L^2| \leq \frac{1}{L^{\frac{\eta}{4}}}.$$

Furthermore, an iteration of Lemma 5.2 as discussed in Proposition 6.1. of [16], we obtain that for $\underline{x} \in \mathcal{B}_L^2$, we have

$$|G_{\underline{x}; \beta, \tilde{\beta}}^{[-L \lfloor \frac{M}{2} \rfloor, L \lfloor \frac{M}{2} \rfloor]}(z; k, \ell)| \leq \frac{1}{2} e^{-\frac{\eta \log(M)}{2M}|k-\ell|}$$

as long as $|k-\ell| \geq \frac{1}{10}L \lfloor \frac{M}{2} \rfloor$. We note that this is still subexponential decay in the size of the interval.

We define L and M by $N = L \lfloor \frac{M}{2} \rfloor$, or equivalently $M = \lfloor \frac{1}{6}N^{\frac{4}{4+\eta}} \rfloor$ and then L as above. Thus, we see that the claim holds for

$$c \in \left(\frac{4}{4+\eta}, 1\right)$$

and N large enough. \square

We define

$$(5.10) \quad \mathcal{U}_{\gamma, \tau, p}(N) = \{\underline{x} : [-N, N] \text{ is not } (\gamma, \tau, p)\text{-suitable for } \mathcal{E}_{\underline{x}} - z\}.$$

With this new notation, we can formulate

Corollary 5.6. *Let $\eta > 1$, $\tau \in (0, 1)$, and $p \geq 1$. There exists $c \in (0, 1)$ such that for $N \geq 1$ large enough, we have for $\gamma = \frac{1}{N^c}$ that*

$$(5.11) \quad |\mathcal{U}_{\gamma, \tau, p}(N)| \leq \frac{1}{N^{\eta}}.$$

Proof. This is an immediate consequence of the previous lemma and Lemma 5.3. \square

We now come to the inductive result

Proposition 5.7. *There exist constants $\eta > 0$, $\sigma > 0$, $\tau \in (0, 1)$, etc
Let $L \geq 1$ be large enough. Then*

$$(5.12) \quad |\mathcal{U}_{\gamma, \tilde{\tau}, 3}(L)| \leq \frac{1}{L^\eta}$$

for some $\tilde{\tau} \in (0, 1)$ implies for $N = L^{\frac{\eta}{2}}$ that

$$(5.13) \quad |\mathcal{U}_{\gamma - N^{\tau-1}, \tau, 3}(N)| \leq e^{-N^\sigma}.$$

Our first goal is to pass from the probabilistic assumption in the proposition to a statement in space.

Lemma 5.8. *There exists $p = p(r) \in (0, 1)$, $\eta > 1$ such that for every $\underline{x} \in \mathbb{T}^r$, $M \geq 1$ large enough, and $N \geq L^{\frac{1}{p^2}}$, we have that*

$$(5.14) \quad \#\{1 \leq n \leq N : T^n \underline{x} \in \mathcal{U}_{\gamma, \tau, 1}(L)\} \leq N^p$$

if

$$(5.15) \quad |\mathcal{U}_{\gamma, \tau, 2}(L)| \leq \frac{1}{N^\eta}.$$

Proof. By Theorem 4.1, we have that there is a semi-algebraic S set of degree $\leq L^C$ and measure $|S| \leq \frac{1}{L^\eta}$ such that $\mathcal{U}_{\gamma, \tau, 3}(L) \subseteq S$. The claim now follows by an application of Theorem A.1. \square

From Lemma 5.3, we now obtain a set $\mathcal{B}_N^{\text{IDS}}$ of measure

$$(5.16) \quad e^{-N^\sigma}$$

such that for every $\underline{x} \in \mathbb{T}^r \setminus \mathcal{B}_N^{\text{IDS}}$ and $-N \leq k < \ell \leq N$, we have

$$(5.17) \quad \|(\mathcal{E}_{\underline{x}; \beta, \tilde{\beta}}^{[k, \ell]} - x)^{-1}\| \leq e^{L^\tau}.$$

Clearly, this implies the first estimate needed for the proof of Theorem 2.5.

Proof of Proposition 5.7. Let $|\ell| \leq \frac{N}{2}$. We will just prove the estimate on $G_{\underline{x}; \beta, \tilde{\beta}}^{[-N, N]}(z; \ell, N)$.

By Lemma 5.8, there exist a minimal $n_{1,+}$ such that $n_{1,+} > \ell$ and

$$|G_{\underline{x}; \beta, \tilde{\beta}}^{n_{1,+} + [-L, L]}(z; n_{1,+}, n_{1,+} \pm L)| \leq \frac{1}{2} e^{-\gamma L}.$$

$n_{1,-}$ is chosen in a similar fashion such that $n_{1,-} \leq \ell$. By Lemma 5.2 and the estimate on the resolvent, we can thus conclude that

$$\begin{aligned} |G_{\underline{x}; \beta, \tilde{\beta}}^{[-N, N]}(z; \ell, N)| &\leq \frac{e^{-(\gamma - L^{\tau-1})L}}{2} \left(|G_{\underline{x}; \beta, \tilde{\beta}}^{[-N, N]}(z; n_{1,+} + L, N)| \right. \\ &\quad + |G_{\underline{x}; \beta, \tilde{\beta}}^{[-N, N]}(z; n_{1,+} - L, N)| \\ &\quad + |G_{\underline{x}; \beta, \tilde{\beta}}^{[-N, N]}(z; n_{1,-} + L, N)| \\ &\quad \left. + |G_{\underline{x}; \beta, \tilde{\beta}}^{[-N, N]}(z; n_{1,-} - L, N)| \right). \end{aligned}$$

We can now choose $n_{2,\pm}$ similarly by requiring $|n_{2,\pm}| \geq |n_{1,\pm}| + L$. Iterating, we find

$$|G_{\underline{x}; \beta, \tilde{\beta}}^{[-N, N]}(z; \ell, N)| \leq 2e^{L^\tau} e^{-SL(\gamma - L^{\tau-1})}$$

for S the maximal time we can perform this operation.

It remains to estimate S . Let $\ell > 0$, then it clearly suffices to estimate the number of times, we can choose $n_{j,+}$. As there are $N - \ell - N^p$ many good points to choose from, and we always need to skip L of these, we see that, we have

$$S \geq \frac{N - \ell - N^p}{L}.$$

This implies $SL \geq |N - \ell| \cdot (1 - \frac{2}{N^{1-p}})$, which clearly implies the claim. \square

Proof of Theorem 2.5. By Corollary 5.6, we can apply Proposition 5.7 inductively to conclude the claim at larger and larger scales. This yields the claim. \square

APPENDIX A. RETURN TIMES ESTIMATES

We know that the skew-shift $T : \mathbb{T}^r \rightarrow \mathbb{T}^r$ is uniquely ergodic. In particular, if $U \subseteq \mathbb{T}^r$ is an open set, we have that for every $\underline{x} \in \mathbb{T}^r$

$$(A.1) \quad \frac{1}{N} \# \{1 \leq n \leq N : T^n \underline{x} \in U\} \rightarrow |U|.$$

The main goal of this section will be to quantify this convergence for special sets called *semi-algebraic*, see Section 4. The methods discussed here can be essentially be extracted from [2].

Theorem A.1. *Let \mathcal{S} be a semi-algebraic set of degree B and ω Diophantine. Then there is $q \in (0, 1)$, $G \geq 1$, and $C \geq 1$ such that for $\underline{x} \in \mathbb{T}^r$*

$$(A.2) \quad \# \{1 \leq \ell \leq L : T^\ell \underline{x} \in \mathcal{S}\} \leq CB^G L^{1-q}$$

as long as

$$(A.3) \quad |\mathcal{S}| \leq \frac{1}{L^{\frac{qr}{2}}}.$$

We will need the following estimate on return times

Theorem A.2. *Let ω be Diophantine and $T_\omega : \mathbb{T}^r \rightarrow \mathbb{T}^r$ be the skew-shift with frequency ω . Then*

$$(A.4) \quad \# \{1 \leq \ell \leq L : T_\omega^\ell \underline{x} \in B_\varepsilon(\underline{a})\} \lesssim \varepsilon^r L + \frac{1}{\varepsilon^r} L^{1-p}$$

for an universal constant $p = p(r) \in (0, 1)$.

In order to pass from balls to semi-algebraic sets, we will need the following result

Theorem A.3. *Let $\mathcal{S} \subseteq \mathbb{T}^r$ be a semi-algebraic set of degree $\deg(\mathcal{S}) \leq B$ for B large enough and measure $|\mathcal{S}| \leq \varepsilon^r$. Then for an universal constant $G \geq 1$, there exists $1 \leq T \leq B^G \varepsilon^{-(r-1)}$ and points x_1, \dots, x_T such that*

$$(A.5) \quad \mathcal{S} \subseteq \bigcup_{t=1}^T B_\varepsilon(x_t).$$

(A.5) states that \mathcal{S} can be covered by less than $\varepsilon^{-(r-1)} B^G$ many ε balls. Here, we denote

$$(A.6) \quad B_\varepsilon(a) = \{x \in \mathbb{T}^K : \|x - a\| \leq \varepsilon\}.$$

Proof of Theorem A.1. Combining these two theorems, we obtain

$$\#\{1 \leq \ell \leq L : T^\ell \underline{x} \in \mathcal{S}\} \lesssim B^G(\varepsilon L + \frac{1}{\varepsilon^{2r-1}} L^{1-p}).$$

Hence, taking $\varepsilon L = \frac{1}{\varepsilon^{2r-1}} L^{1-p}$ or equivalently

$$\varepsilon = \frac{1}{L^{\frac{p}{2r}}}$$

we obtain that

$$\#\{1 \leq \ell \leq L : T^\ell \underline{x} \in \mathcal{S}\} \lesssim B^G L^{1-\frac{p}{2r}}$$

which is the claim. \square

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